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Flow Networks and Fuzzy-State Systems

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Abstract

We present a generalization of the results in [8] concerning a generalization of basic concepts in classical crisp binary and multistate models in Reliability Theory where fuzzy numbers are used to describe the space of states. In particular, we show that in our formalization classical flow networks algorithms can be used to find the maximum level of performance possible for a specific system subject to several linguistic constraints.

Key words: Reliability Theory, Structure Functions, Fuzzy Numbers.

1 Introduction.

Description of any given system in terms of its performance is a key step in order to understand such a system and therefore in the search for improvement in the quality of the system. In Reliability Theory, such a description is modeled according to a so called *structure function*. This structure function is just a mapping telling us the performance of the system depending on the performance of each one of its components.

The structure function is modeling a deterministic real system under study, at some fixed time. All static properties of the system are fully explained. The structure function mapping assigns a system state to each possible profile of component states, and any posterior study will refer to such a mathematical representation of the real system. It is therefore assumed in this mathematical model that the state of the whole system is known from the states of all its components, so we can analyze how changes in the component performances modify the system performance.

In classical Reliability Theory (see, e.g., [1]) the basic assumption is that the performance of the system must be in one of two extreme states: either perfect functioning or complete failure (usually denoted by "1" and "0", respectively). The same binary assumption is made for every component. No intermediate state is allowed. In order to apply classical Reliability Theory, practitioners need to model real systems by means of those two extreme states, both for the system and each one of its components.

Since real systems use to be complex systems, it is clear that such a binary assumption relative to performace states will be too restrictive for many real applications. Few real simple systems can be found being always in one of those two extreme states, with no change in performance till they brake down. Most real system not only age because the probability of an unacceptable output increases with time, but because the quality of the output decreases.

Different alternative approaches have been proposed in the past in order to allow intermediate states between perfect functioning and complete failure.

In particular, a general model was considered in [19] by analyzing structure functions

$$\psi : L^n \to L_0$$

where L and L_0 are two arbitrary complete lattices, and n is the number of components (see also [5,18,21]).

If $L = L_0 = \{0, 1\}$, we have the classical *binary* systems.

If $L = L_0 = \{0, 1, \ldots, K\}$, we have the *multistate* systems (see a survey in [11]).

If $L = L_0 = [0, 1]$ or any other compact real interval, we talk about *continuum* systems ([2,3,20]). Continuum systems, that is, those that can be represented by means of a structure function

$$\psi : [0,1]^n \to [0,1],$$

do allow a *gradation* between perfect functioning 1 and complete failure 0. But sometimes it is not clear at all which one is the true state. Such an uncertainty about the real number to be chosen may of course be, in some occasions, of a probabilistic nature. Some other times, fuzziness is more appropriate in order to model such an uncertainty. In other words, it may happen that the state is "1" to some extent and "0" to some other extent, in the sense of a *fuzzy number*.

A fuzzy generalization of continuum systems, where state space $L = L_0$ is an appropriate family of fuzzy numbers defined in the closed unit interval has been introduced in [8].

2 Fuzzy-state structure functions.

As pointed out above, fuzzy-state assumptions are justified when the system or its components are simultaneously to some extent in a fuzzy success and in a fuzzy failure state. Such a situation is common when dealing with real systems. States are not easily ranked on the real line. Defining a crisp valuation set may not be clear at all, and such a difficulty may be due to some inherent fuzziness. One alternative to crisp states in the unit interval is the use of fuzzy numbers. This approach can be very useful in order to deal with linguistic variables (see, e.g., [23]).

A fuzzy number is defined by a membership function $\mu : \Re \to [0,1]$ and a standard restriction is to assume that μ is *normal* and *convex* (see, e.g., [15]), that is,

- there exists a *mean value* $m \in \Re$ such that $\mu(m) = 1$, and
- $\mu(y) \geq \min\{\mu(x), \mu(z)\}$ whenever $x < y < z$.

We shall denote from now on by $\mathcal{N}$ the set of all normal and convex fuzzy numbers in the closed unit interval.

Since the fuzzy numbers in $\mathcal{N}$ are defined on a compact interval (as the closed unit interval), it has been in fact assumed that both the concept of perfect functioning and the concept of complete failure are clearly defined. Obviously, any real number in $[0,1]$ is an element of $\mathcal{N}$ as well.

Given two fuzzy numbers

$$\mu, \nu : \Re \to [0,1]$$

the least upper bound $\mu \vee \nu$ is defined as the fuzzy number

$$(\mu \vee \nu)(t) = \sup_{\max(x,y)=t} \min\{\mu(x), \nu(y)\}$$

and the greatest lower bound $\mu \wedge \nu$ is given by

$$(\mu \wedge \nu)(t) = \sup_{\min(x,y)=t} \min\{\mu(x), \nu(y)\}.$$

The usual partial order on the set of fuzzy number has in this way defined. A fuzzy number μ is then smaller than or equal to another fuzzy number ν (that is, $\mu \leq \nu$) whenever $\mu \vee \nu = \nu$ (equivalently, $\mu \wedge \nu = \mu$). A partial order relation $\leq$ has been therefore defined on the set of fuzzy numbers. In particular, normal convex fuzzy numbers form a distributive lattice under the above standard "$\vee$" (sup, join) and the "$\wedge$" (inf, meet) operations (see Mizumoto-Tanaka [16,17]). Dealing with normal convex fuzzy numbers in the closed unit interval allows as well the evaluation of least upper and greatest lower bounds for arbitrary families of fuzzy numbers (see [8]). Moreover, both least upper bound and greatest lower bound are convex and normal. Therefore, the set $\mathcal{N}$ of all normal convex fuzzy numbers in the closed unit interval is a complete lattice.

In our context, given a fuzzy number $\mu \in \mathcal{N}$, the value $\mu(x)$ will represent the degree to which each number $x \in [0,1]$ is appropriate in order to explain the true state.

In order to avoid computational problems, approximate solutions to some problems can be tried by considering a particular family in $\mathcal{N}$. For example, the family of normal *triangular* or *trapezoidal* fuzzy numbers in the unit interval (see, e.g., [15]). These fuzzy numbers are continuous in such a way that small measurement errors in the membership function do not lead to big changes. For sake of completeness, it should be pointed out the existence in the fuzzy literature of several definitions of what a *fuzzy number* is (see [14]).

From now on we shall assume that our space of states is the family $\mathcal{N}$ of normal convex fuzzy numbers in the unit interval. In particular, any mapping $\phi : \mathcal{N}^n \to \mathcal{N}$ will be called a *fuzzy-state structure function*.

As in classical Reliability Theory, we can focus our attention on those systems which are *isotone*, in the sense that their associated structure function is nondecreasing with respect to each coordinate. That is, those structure functions such that system performance does not improve when component performances get worse. Obviously, such an isotonicity (nondecreasingness) depends on the particular partial order associated to the space of states. In this paper we shall assume the previously introduced partial order on $\mathcal{N}$. Alternative partial orders can be considered (see, e.g., [4]), but they should be in principle consistent with the standard partial order on $\mathcal{N}$.

We conclude the section with our definition of isotonicity.

DEFINITION 1 *A fuzzy-state structure function ϕ is* isotone *if it is nondecreasing in each coordinate, that is,*

$$\phi(\mu_1, \ldots, \mu_n) \leq \phi(\mu'_1, \ldots, \mu'_n)$$

whenever $\mu_i \leq \mu'_i$ for all i.

3 Fuzzified continuum systems.

An interesting type of fuzzy-state structure functions can be obtained by applying Zadeh's extension principle [26] to crisp continuum systems (see also [9,24] and [7]). Zadeh's extension principle has been already applied in the past in order to fuzzify probabilistic information in reliability, for example by defining performance as a fuzzy number in the unit real interval where probabilities take values (see [13,22]).

Let us consider now a *coherent* continuum system, that is, a system modeled according to a isotone structure

function

$$\psi : [0,1]^n \to [0,1]$$

such that $\psi(1,1,\ldots,1) = 1$ and $\psi(0,0,\ldots,0) = 0$. We shall assume here that such a mapping ψ is also a continuous mapping. Then we define its associated fuzzy-state structure function as the mapping

$$\phi_\psi : \mathcal{N}^n \to \mathcal{N}$$

such that

$$\phi_\psi(\mu_1, \ldots, \mu_n)(t) = \sup_{\psi(x_1,\ldots,x_n)=t} \min\{\mu_1(x_1), \ldots, \mu_n(x_n)\}$$

Such a mapping ϕ_ψ is always well defined within our context.

THEOREM 1 *Given $\psi : [0,1]^n \to [0,1]$ coherent and continuous, then $\phi_\psi(\mu_1, \ldots, \mu_n) \in \mathcal{N}$, for all $\mu_1, \ldots, \mu_n \in \mathcal{N}$.*

In addition, the fuzzy-state structure function ϕ_ψ associated to a continuous coherent continuum structure function ψ is assured to be *coherent* whenever ψ is *coherent* in the above sense. In fact, every associated fuzzy-state structure function verifies isotonicity with respect to the partial order of fuzzy numbers in the closed unit interval, and perfect functioning (or complete failure) will appear whenever every component is in the perfect functioning state (complete failure, respectively).

THEOREM 2 *Given $\psi : [0,1]^n \to [0,1]$ coherent and continuous, then ϕ_ψ is isotone with respect to the usual partial order of fuzzy numbers. Moreover, if $x_1, \ldots, x_n \in [0,1]$ then*

$$\phi_\psi(x_1, \ldots, x_n)(t) = \psi(x_1, \ldots, x_n)$$

4 Fuzzy-state series-parallel systems.

As in classical Reliability Theory, an interesting question to be solved is which components should be improved in performance in order to improve the system performance. Taking Birnbaum's structural importance for crisp binary systems (see, e.g., [1]), we propose the following definition.

DEFINITION 2 *Given a fuzzy-state structure function ϕ_ψ with ψ its associated continuum structure function, and the states $\mu_i \in \mathcal{N}$ for each component i being fixed, a critical set of components is any minimal subset of components C such that*

$$\phi_\psi(\mu_1, \ldots, \mu_n) < \phi_\psi(\nu_1, \ldots, \nu_n)$$

for some $\nu_1, \ldots, \nu_n \in \mathcal{N}$ such that

(a) $\nu_i > \mu_i$ for all $i \in C$;

(b) $\nu_i = \mu_i$ if $i \notin C$;

(c) $\nu_i < \mu_j$ for all $i \in C$ and any j such that $\mu_i < \mu_j$.

That is, a critical set is any minimal subset of components such that any arbitrarely small improvement of their performance implies an improvement of the system performance. In particular, condition (c) avoids trivial solutions and clarifies the meaning of "small" improvement. Being minimal means that there is no proper subset of C verifying such a property. In fact, we shall not be interested in improving component performance if the system performance remains constant. Of course, such a critical set may not be unique. Moreover, it may happen that no improvement is possible (i.e., the system performance is already 1), and in this case the critical set is empty.

Searching for such critical sets involve in general hard computational problems. An important practical case in which the solution can be easily obtained has been analyzed in [6] under the following two assumptions:

(i) the particular family of fuzzy numbers representing the state of each component is linearly ordered.

(ii) the associated crisp system ψ defines a series-parallel system;

Notice that the use of linguistic performance terms like small, medium, large or few, many, not many, etc. is quite usual in the fuzzy-setting (see, e.g., [22]). These terms are linearly ordered and can be shared by all the components.

Series-parallel configuration appears in many real systems (see, e.g., [10]). Their structure function is some combination of the components in series and parallel compositions, which are respectively associated to min and max operators. A series-parallel continuum system is in fact a Barlow-Wu continuum system (see, e.g., [25]), and it can be therefore characterized by means of an underlying crisp binary $\{0,1\}$-system. In particular, these series-parallel configurations can be represented in terms of a binary tree whose vertices are components and modules, being the root of this tree the system itself. Being n the number of components, such a binary tree shows how the state of the system is to be evaluated, step by step from the previous evaluation of every subsystem, by means of $n-1$ binary operators, either series or parallel. Each vertex in such a binary tree representation has two (and no more) descendent elements, in such a way that either the maximum state or the minimum state among those two descendent elements is assigned to it. Moreover, each component appears only once in this binary tree.

In other words, the series-parallel system can be described by

- $n-1$ modules: $\{M_{n+1}, \ldots, M_{2n-1}\}$, being M_{2n-1} the system itself;
- $n-1$ operators: $O(j) = s$ if module M_j has series configuration, or $O(j) = p$ if it has parallel

configuration, and

- $n-1$ pairs $\{F(j), S(j)\}$ denoting the two elements of module M_j, chosen in such a way that $F(j), S(j) \in \{1, \ldots, j-1\}$.

Being our crisp basis system ϕ a series-parallel structure, in case the component performance states are linearly ordered it is easy to check that the system performance will be the performance of all components within any critical set of components, if there exists any (otherwise, the system performance is already 1 and can not be improved).

THEOREM 3 *Let $\psi : [0,1]^n \to [0,1]$ be a series-parallel continuum system, and let us assume that the components of the associated fuzzy-state system ϕ_Ψ can be ordered in such a way that their performances are nondecreasing, i.e.,*

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n.$$

Then the system performance coincides with the performance of every component in any critical subset of components, if any.

In order to improve a series-parallel system with linearly ordered component performances, we can use the above theorem. An algorithm to compute a critical subset of components, depending on the required improvements at each stage, has been given in [8]. We remark that although a small improving in the performance of all components in any critical set will improve the system, such an improvement is not assured for bigger performance improvements. In fact, a critical set in a certain case may not be still a critical set once its components have been modified in performance.

5 Fuzzy-State Weigted Systems

As it is often the case, real life suggests several ways to improve mathematical results. In our case, we may very easily envisage a system were the components connections are such that max and min are no longer valid mathematical means for computing the system performance. Moreover, the system components may be of importance (to different degrees) for more than one module and some module performances have to be considered as constant, that is to say not improvable, because of specific reasons. For instance, if the system under study is a running car, we may want to study its level of performance under different fixed conditions which may be low speed, medium speed or high speed, etc. Speed may be seen as the level of performance of the module "engine", however, we do not necessarily want to bring the module "engine" to its maximum level of performance. Summing up, we can formalize as follows:

- m modules: $\{M_1, \ldots, M_m\}$, being M_m the system itself;
- a System Network (Directed Graph) $\mathcal{G}$ whose nodes are the modules and the basic components;
- the edges of $\mathcal{G}$ are weighted in such a way that for any module M_k, if $w_1, \ldots, w_{j_k}$ are the weights of the edges entering M_k, then $\sum_{i=1}^{j_k} w_i = 1$.
- the degree of performance of each module M_k is given by

$$\sum_{i=1}^{j_k} w_i p_i$$

with $p_1, \ldots, p_{j_k} \in \mathcal{N}$ being the performances of the modules or basic components connected to M_k;

- an operability degree $o_k \in \mathcal{N}$ for each module M_k denoting the maximum level of performance possible or "recommended" for that module.

Notice that

1. the graph $\mathcal{G}$ may have cycles, i.e. the level of performance of the modules may be inter-related.
2. $\mathcal{N}$ is closed with respect to multiplication by a constant and the sum operator. In particular, triangular fuzzy numbers are closed with respect to these operators (see [15]).

The problem we have is now the following: how do we compute the maximum level of performance which can be reached by M_m ?

To answer the above question we rewrite our problem as a Network Flow Problem (see [6]). If we are able to accomplish that then not only we would have a clear solution but efficient algorithms as well.

6 The network flow problem

Let us briefly describe what is the network flow problem.

A network is a directed graph with a real number $cap(e)$ assigned to each edge e in the graph: $cap(e)$ is called *capacity* of the edge. The network has two distinguished vertices the "source" and the "sink". A flow in the network is a function f which assigns to each edge e different than the source and the sink a real value $0 \leq f(e) \leq cap(e)$ such that for each vertex v

$$\sum_{e \in in(v)} f(e) = \sum_{e \in out(v)} f(e)$$

where $in(v)$ and $out(v)$ are, respectively, the sets of edges entering and exiting the vertex v.

In order to find efficient algorithms for finding the maximum flow which can go through the network, it is useful to define the notion of a "cut". A cut in the network is a partition $(W, \bar{W})$ of the set of vertices V, such that the source is in W and the sink is in $\bar{W}$. Thus, since $\bar{W} = V \setminus W$, it is clear that any flow that

leaves the source and it reaches the sink must at some instant flow from a vertex of W to a vertex of $\bar{W}$, i.e. an edge of the cut. If we define the capacity of a cut as the sum of all the flow that goes through the edges of the cut, then we have the "max-flow min-cut" theorem (see also [12]).

THEOREM 4 *The maximum possible value of any flow in a network is equal to the minimum capacity of any cut in that network.*

It is clear that the cut in a network plays the same role as the critical set of components in our case. At any rate, without getting too much in details (which will be analyzed in an extended version of this paper) we can see that the sought re-formalization of the maximum performance problem is in some sense quite straightforward:

- We add an extra node S to $\mathcal{G}$. Such a node, the "source", will be connected to all the basic components of the system and only to them;
- the node corresponding to the system M_m will be the sink;
- for each vertex M_k the operability degree o_k of the corresponding module or component, will be the "capacity" degree of all the edges exiting the node.

The maximum flow of the corresponding network will be the maximum level of performance allowed for our system.

7 Final comments.

Fuzzy-state structure functions can be useful as an alternative to model uncertainty of real systems.

We have shown in this paper that the fuzzy-state structure function when applied to some specific systems, give rise to a problem of performance improving which can be transformed into an equivalent network flow problem. As a consequence, we obtain fast algorithms for computing a critical set of component and the maximal allowed performance.

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Different Types of Measures of Fuzzy Sets and Their Integral Representation

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Abstract

We start with measures of crisp sets (including some generalizations of the additivity property) and extend them to the fuzzy case using appropriate integration concepts, thus obtaining different types of measures of fuzzy sets. For important cases of such measures axioms as well as integral representations are presented.

1 MEASURES OF CRISP SETS

In classical measure theory, the starting point is usually a *measurable space* $(X, \mathcal{A})$, where $\mathcal{A}$ is a σ-algebra of subsets of the universe X, and a *measure* m [1] is defined as a function $m : \mathcal{A} \to [0, +\infty]$, satisfying some *boundary conditions*, most typically

$$m(\emptyset) = 0, \tag{1}$$

the *additivity*

$$A \cap B = \emptyset \Longrightarrow m(A \cup B) = m(A) + m(B), \tag{2}$$

and, in the case of an infinite universe X, a *continuity property*, e.g., *continuity from below*

$$(A_n)_{n\in\mathbb{N}} \nearrow A \Longrightarrow \lim_{n\to\infty} m(A_n) = m(A), \tag{3}$$

or *continuity from above*

$$(A_n)_{n\in\mathbb{N}} \searrow A \Longrightarrow \lim_{n\to\infty} m(A_n) = m(A). \tag{4}$$

In our context, we shall call a measure satisfying all the properties (1)–(4) a *σ-additive measure*.

Most generalizations of this classical concept concern the additivity property (boundary condition and continuity are either kept unchanged or play a minor role). From our point of view, these are the most interesting ones (in chronological order):

(i) *Fuzzy measures:* In his thesis, Sugeno [26] required a fuzzy measure m to satisfy the boundary conditions (1) and $m(X) = 1$, both the continuity from below (3) and from above (4) and, instead of the ditivity (2), the weaker *monotonicity*

$$A \subseteq B \Longrightarrow m(A) \leq m(B).$$

(ii) *Possibility measures:* A possibility meas (Zadeh [34]) satisfies the boundary conditions (1) a $m(X) = 1$, the continuity from below (3) and for a sets $A, B \in \mathcal{A}$

$$m(A \cup B) = \max(m(A), m(B)).$$

(iii) *Decomposable measures:* Given a t-conorm (see [25] and Section 2 below), a $\perp$-decomposable me sure m (Dubois [8], Weber [28]) satisfies the boundar conditions (1) and $m(X) = 1$, the continuity from b low (3) and for all $A, B \in \mathcal{A}$ the *decomposition pro erty*

$$A \cap B = \emptyset \Longrightarrow m(A \cup B) = m(A) \perp m(B).$$

Obviously, if $\perp = S_M$, we obtain exactly a possibilit measure as a special case.
In the case of an *Archimedean* t-conorm $\perp$ with *ad ditive generator* g (see [25]) there are two importan types of $\perp$-decomposable measures m [20]:

(A) $g \circ m$ is a σ-additive measure,

(P) there is a sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint elements of $\mathcal{A}$ such that

$$(g \circ m)\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = g(M) < \sum_{n\in\mathbb{N}} (g \circ m)(A_n).$$

2 TRIANGULAR NORMS AND CONORMS

A *triangular norm* (t-*norm* for short, see Schweizer & Sklar [25]) T is an associative, commutative operation on $[0, 1]$ which is isotone in both components and has 1 as neutral element. The dual t-*conorm* S is then given by

$$S(x, y) = 1 - T(1 - x, 1 - y).$$

Triangular norms and conorms are used in fuzzy set theory to model intersection and union, respectively.

PROCEEDINGS

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